

# HOW MANY STATISTICS ARE NEEDED TO CHARACTERIZE THE UNIVARIATE EXTREMES

GANE SAMB LO

**ABSTRACT.** Let  $X_1, X_2, \dots$  be a sequence of independent random variables ( $rv$ ) with common distribution function ( $df$ )  $F$  such that  $F(1) = 0$ . We consider the simple statistical problem : find a statistics family of size  $m \geq 1$  whose convergence, in probability or almost surely, to a point of some domain  $\mathcal{S} \in \mathbb{R}^m$  is equivalent that  $F$  lies in the extremal domain of attraction  $\Gamma$ . Such a family, whenever it exists, is called an Empirical Characterizing Statistics Family for the EXTremes (ECSFEXT). The departure point of this theory goes back to Mason [24] who proved that the Hill ([18]) estimator converges a.s. to a positive real number for some particular sequences if and only  $F$  lies in the attraction domain of a Frchet's law. Considered for the whole attraction domain, the question becomes more complex. We provide here an ECSFEXT of nine (9) elements and also characterize the subdomains of  $\Gamma$ . The question of lowering  $m=9$  to a minimum number is launched.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN PROBLEM

Let  $X_1, X_2, \dots$  be a sequence of independent and identically associated with the  $df$   $F$ , with  $F(1) = 0$  and let for once  $G(y) = F(e^y)$  an auxilliary  $df$  associated with independent and identically distributed random variables  $\log X_1, \log X_2, \dots$ . For each  $n \geq 1$  fixed, their order statistics are denoted by

$$X_{1,n} = \log Y_{1,n} \leq X_{2,n} = \log Y_{2,n} \leq \dots \leq X_{n,n} = \log Y_{n,n}.$$

The departure problem of Univariate Extreme Value Theory (UEVT) is finding the asymptotic law of the maximum observation  $X_{n,n} = \max(X_1, \dots, X_n)$ . In this theory, the  $df$   $F$  is said to be attracted to a non degenerated extremal  $df$   $M$  iff the maximum  $X_{n,n} = \max(X_1, \dots, X_n)$ , when appropriately centered and normalized

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by two sequences of real numbers  $(a_n > 0)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$ , converges to  $M$ , in the sense that

$$(1.1) \quad \lim_{n \rightarrow +\infty} \mathbb{P}(X_{n,n} \leq a_n x + b_n) = \lim_{n \rightarrow +\infty} F^n(a_n x + b_n) = M(x),$$

for continuity points  $x$  of  $M$ . If (1.1) holds, it is said that  $F$  is attracted to  $M$  or  $F$  belongs to the domain of attraction of  $M$ , written  $F \in D(M)$ . It is well-known that the three nondegenerate possible limits in (1.1), called extremal  $df$ 's, correspond to three possibilities are the following.

The Gumbel  $df$

$$(1.2) \quad \Lambda(x) = \exp(-\exp(-x)), \quad x \in \mathbb{R},$$

the Fréchet  $df$  with parameter  $\gamma > 0$

$$(1.3) \quad \phi_\gamma(x) = \exp(-x^{-\gamma}) \mathbb{I}_{[0, +\infty[}(x), \quad x \in \mathbb{R}$$

and the Weibull  $df$  with parameter  $\beta > 0$

$$(1.4) \quad \psi_\beta(x) = \exp(-(x)^\beta) \mathbb{I}_{]-\infty, 0]}(x) + (1 - \mathbb{I}_{]-\infty, 0]}(x)), \quad x \in \mathbb{R},$$

where  $\mathbb{I}_A$  denotes the indicator function of the set  $A$ .

Actually the limiting  $df$   $M$  is defined by an equivalence class of the binary relation  $\mathcal{R}$  on the set  $\mathcal{D}$  of  $cdf$ 's on  $\mathbb{R}$ , defined as follows

$$\begin{aligned} \forall (M_1, M_2) \in \mathcal{D}^2, (M_1 \mathcal{R} M_2) &\Leftrightarrow \exists (a, b) \in \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}, \forall (x \in \mathbb{R}) : \\ M_2(x) &= M_1(ax + b). \end{aligned}$$

One easily checks that if  $F^n(a_n x + b_n) \rightarrow M_1(x)$ , then  $F^n(c_n x + d_n) \rightarrow M_1(ax + b) = M_2(x)$  whenever

$$(1.5) \quad a_n/d_n \rightarrow a \text{ and } (b_n - d_n)/c_n \rightarrow b \text{ as } n \rightarrow \infty.$$

These facts allow to parameterize the class of extremal  $df$ 's. For this purpose, suppose that (1.1) holds for the three  $df$ 's given in (1.2), (1.3) and (1.4). If we take sequences  $(c_n > 0)_{n \geq 1}$  and  $(d_n)_{n \geq 1}$  such that the limits in (1.5) are  $a = \gamma = 1/\alpha$  and  $b = 1$  (in the case of Fréchet extremal domain), and  $a = -\beta = -1/\alpha$  and  $b = -1$  (in the case of Weibull extremal domain), and finally, if we interpret  $(1 + \gamma x)^{-1/\gamma}$  as  $\exp(-x)$  for  $\gamma = 0$  (in the case of Gumbel extremal domain), we are entitled to write the three extremal  $df$ 's in the parameterized shape

$$(1.6) \quad G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad 1 + \gamma x \geq 0,$$

called the Generalized Extreme Value (GEV) distribution function with parameter  $\gamma \in \mathbb{R}$ .

For a modern and large account of the Extreme Value Theory, the reader is referred to Beirlant *et al.* [1], Galambos [16], de Haan [6], de Haan and Ferreira [5] and Resnick [26].

The problem of estimating the extremal index  $\gamma$  by various and numerous estimators and finding statistical tests based on those estimators has been extremely

widely tackled by many authors in papers and books. Let us only cite a sample of these authors as : Pickands [25], Hall (1981) [17], Berilant and Teugels (1986) [2], Deheuvels and Mason (1990) [9], Deheuvels and Mason (1990) [10], Deheuvels, Haeusler and Mason (1988) [8], Csörgő, Haeusler and Mason [4] and Lô [19], [20] etc. Even in these last years new statistics continue to appear in the frame of new methodologies such as adaptative procedures and second and third order condition, etc.

This paper is not only about statistical estimation of the extremal domain, in the sense that the convergence of some statistics  $S_n$  to a function of the extremal index  $g(\gamma)$ , under the hypothesis (H) that  $F$  lies in  $\Gamma$ , yields a statistical test of (H) with  $(|S_n - g(\gamma)| > c)$  as a rejection region. We also face the inverse question : does the convergence of  $S_n$  to  $g(\gamma)$  implies that (H) holds. This is the empirical characterization problem that we set and motivate in the next Section 2. In Section 3, we give a general solution proved in Section 4. Concluding remarks end the paper in Section 5.

## 2. THE PROBLEM AND ITS MOTIVATION

We are now describing the Mason fundamental result which is the departure point of our question. Consider a sequence of integers  $k = k_n$ ,  $n \geq 1$  satisfying,

$$(K) \quad k_n \rightarrow \infty \text{ and } k_n/n \rightarrow 0 \text{ as } n \rightarrow \infty$$

and consider for  $0 < \alpha < 1$ ,  $k_n(\alpha) = [n^\alpha]$ , where  $[x]$  denotes the integer part of  $x$ , that is the unique integer such that  $[x] \leq x < [x] + 1$ , special cases of sequences satisfying (K). Mason [24] (1982) proved the following.

**Theorem 1.** *For any  $0 < \gamma < \infty$ , and  $\ell = 1$ ,  $F \in D(\phi_\gamma)$  if and only if*

(i) *for some  $0 < \alpha < 1$ ,*

$$T_n(2, k, \ell) = k^{-1} \sum_{i=\ell}^{i=k} i(\log X_{n-i+1, n} - \log X_{n-i, n}) \rightarrow 1/\gamma, p.s.$$

*as  $n \rightarrow +\infty$*

(ii) *if and only if for all sequences satisfying (K),*

$$T_n(2, k, \ell) \rightarrow_P 1/\gamma$$

*as  $n \rightarrow +\infty$ .*

This is the first step of what we call empirical characterizations of the extremes achieved only with the Hill statistic  $T_n(2, k, \ell)$ . From this, we formulate the following general problem.

Given only the observations  $X_1, X_2, \dots$  associated with an unknow underlying  $df$   $F$ , is it possible to answer these three questions ? First

( $\mathcal{P}$ ) : Is it possible to find a set of statistics, that is a vector of  $m \geq 1$  statistics  $S_n = (S_n(1), \dots, S_n(m))$  and a subset  $\mathcal{S}$  of  $\mathbb{R}^m$  that such the convergence of  $S_n$  to a point of  $\mathcal{S}$  is a necessary and sufficient condition for  $F$  to ly in the extremal domain  $\Gamma$ ?

This problem may be rephrased as follows : Is it possible to demonstrate the existence of  $(S_n)$  and  $\mathcal{S}$  such that :

$$(2.1) \quad (F \in \Gamma) \iff \exists(a \in \mathcal{S}), (S_n \rightarrow a)$$

where the limit is almost sure or in probability.

We denote this as a **global** empirical characterization of the extremal domain. The statistics  $S_n$ , if it exists will be called an Empirical Characterizing Statistics Family for the Extremes (ECSFEXT).

Let us introduce this notation. For  $\mathcal{S}$  of  $\mathbb{R}^m$ , with  $m \geq 1$ , we call  $\Pi(\mathcal{S})$  the set all projections of  $\mathcal{S}$  on its components.

If this question is positively answered, we go further and find to search to partition  $\mathcal{S}$  into three subdomains  $\mathcal{S}_0$ ,  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , the two latters being parameterized by  $\gamma > 0$ , that this

$$(2.2) \quad S_1 = \{a(\gamma), \gamma > 0\}, S_2 = \{b(\gamma), \gamma > 0\}$$

such that there exists  $\pi \in \Pi(\mathcal{S})$  so that

$$(2.3) \quad (F \in \Lambda) \iff \exists(a \in S_0), (\pi(S_n) \rightarrow \pi(a)),$$

for any  $\gamma > 0$ ,

$$(2.4) \quad (F \in D(\phi)) \iff \{\exists(\gamma > 0), (\pi(S_n) \rightarrow \pi(a(\gamma))) \in S_1\} \Rightarrow (F \in D(\phi_\gamma))$$

and for any

$$(2.5) \quad (F \in D(\psi)) \iff \{\exists(\gamma > 0), (\pi(S_n) \rightarrow \pi(b(\gamma))) \in S_2\} \Rightarrow (F \in D(\phi_\gamma)).$$

When the empirical characterization concerns any particular case (2.3), (2.4) or (2.5), we qualify it as **simple**.

At this point, Mason [24] have solved the case (2.4) in a very general way, both in probability limits and in almost sure limits.

We should not be confusing this empirical characterization problem with that of the estimation or the selection of the extremal domain. For this, we have :

**Definition 1.** A family of  $m$  statistics  $S_n$  is an *Estimating Statistics Family for the Extremes (ESFEXT)* if there exists a subset  $\mathcal{S}$  of  $\mathbb{R}^m$  partitionned into  $\mathcal{S}_0$ ,  $\mathcal{S}_1$  and  $\mathcal{S}_2$  where  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are parameterized as in (2.2), such that

$$(2.6) \quad (F \in \Lambda) \implies S_n \rightarrow a \in S_0,$$

for any  $\gamma > 0$ ,

$$(2.7) \quad (F \in D(\phi_\gamma)) \implies (S_n \rightarrow a(\gamma)) \in S_1$$

and for any  $\gamma > 0$

$$(2.8) \quad (F \in D(\psi_\gamma)) \implies (S_n \rightarrow b(\gamma)) \in S_2$$

This problem will be addressed in the next section.

## 3. A GENERAL SOLUTION

Define the following statistics

$$(3.1) \quad A_n(1, k, \ell) = k^{-1} \sum_{j=\ell+1}^{j=k} \sum_{i=j}^{i=k} i \delta_{ij} (Y_{n-i+1, n} - Y_{n-j+1, n}) (Y_{n-j+1, n} - Y_{n-j, n}),$$

where  $i \delta_{ij} = \frac{1}{2}$  if  $i = j$  and  $\delta_{ij} = 1$  if  $i \neq j$  ( $k, \ell$ ), is a couple of integers such that  $1 \leq \ell < k < n$ ,

$$y_0 - Y_{n-k, n}, 1 < k < n$$

when  $x_0(G) = y_0 < +\infty$ . From these two statistics and from  $T_n(2, k, \ell)$ , we form our ECSFEXT. Before we go any further, we should remark that  $A_n(1, k, \ell)$  was new in 1989. We discovered later that is related to that of de Dekkers et al. [11] (1989)

$$(3.2) \quad M_n^{(2)}(k) = \frac{1}{k} \sum_{j=1}^k (Y_{n-k+1, 1} - Y_{n-k, n})^2,$$

in the sense that  $A_n(1, k, 1) = 2M_n^{(2)}(k)$ . We establish this in Subsection 7.2 of the Appendix Section 7. We shall use this remark to rediscover the result of de Dekkers *et al.* [11] in new ways. Here is our ECSFEXT

$$S_n = {}^t(T_n(1), \dots, T_n(9)) \in \mathbb{R}^9,$$

where

$$T_n(2) = T_n(2, k, \ell),$$

$$T_n(1, k, \ell) = T_n(2, k, \ell) A_n(1, k, \ell)^{-1/2},$$

$$T_n(5) = T_n(2, \ell, 1),$$

$$T_n(6) = T_n(2, \ell, 1) / (Y_{n-\ell, n} - Y_{n-k, n}),$$

$$T_n(7) = A_n(1, \ell, 1) / (Y_{n-\ell, n} - Y_{n-k, n})^2,$$

$$T_n(8, v) = n^{-v} (Y_{n-\ell, n} - Y_{n-k, n})^{-1}$$

and, when  $Y_{n, n} \uparrow y_0 < +\infty$ ,

$$T_n(9) = (y_0 - Y_{n-\ell, n}) / (y_0 - Y_{n-k, n}).$$

Finally put

$$\mathcal{S} = \mathbb{R}_+^2 \times \{0\}^2 \times \bar{\mathbb{R}}_+ \times \mathbb{R}_+ \times \{0\}^2 \subset \mathbb{R}^9.$$

We denote by  $\pi_{p, n}$ , the projection of  $\mathbb{R}^n$  onto  $\mathbb{R}^p$  when  $p < n$ . We begin to state the estimation of the extremal domain.

**Theorem 2.** *Let  $F \in \Gamma$ , then for all sequences  $k = k(n) = [n^\alpha]$ ,  $\ell = [n^\beta]$ ,  $\frac{1}{2} < \beta < \alpha < 1$ , for any  $v > 0$ ,*

- (i)  $\pi_{4, 9}(S_n)$  converges almost surely to some  $\pi_{4, 9}(A)$ ,  $A \in \pi_{4, 9}(\mathcal{S})$ , as  $n \rightarrow +\infty$ .

- (ii)  $\pi_{7,9}(S_n)$  converges in probability to some  $A \in \pi_{4,9}(\mathcal{S})$ . Specifically,
  - (ii-a) If  $F \in D(\Lambda)$ , then  $A = (1, 0, 0, y_0, 0, 0, 0)$ ;  $y_0 = +\infty$  or  $y_0 < +\infty$ .
  - (ii-b) If  $F \in D(\phi_\gamma)$ , then  $A = (1, \gamma^{-1}, 0, +\infty, \gamma^{-1}, 0, 0)$ , for  $\gamma > 0$ .
  - (ii-c) If  $F \in D(\psi_\gamma)$ , then  $A = [1 - (2 + \gamma)^{-1}]^{\frac{1}{2}} (0, 0, y_0, 0, 0, 0)$ ; for  $y_0 = +\infty, \gamma > 0$ .
- (iii) In addition,
  - 4) If  $F \in D(\Lambda) \cup D(\Phi)$ , then  $T_n(8) \rightarrow 0$ , a.s., as  $n \rightarrow +\infty$ .
  - 5) If  $D(\psi_\gamma)$ , then  $T_n(9) \rightarrow 0$ , a.s., as  $n \rightarrow +\infty$ .

**Remark 1.** At this stage we see that the couple  $(T_n(1), T_n(2))$ , and then the couple  $(A_n, T_n(2))$ , suffices to estimate the whole domain of attraction. One would like to have it as an ECSFEXT. Unfortunately, we need more other statistics to achieve the full empirical characterization in Theorem 3 below.

**Remark 2.** It would be legitimate to ask whether other simple statistics have this empirical characterization property for the Frechet domain of attraction as the Hill one does. At this stage, we simply remark that multiples and one-to-one functions of Hill's statistic surely inherit this property. It is the case for our statistic (3.1) and the de Dekkers et al. (3.2) moment estimator. As for the de Haan and Resnick (7.1) statistic, which is a estimator of the extremal index when this latter is positive, does not possess this property as we show it in Subsection 7.1 of Section 7.

By inverting the above theorem in the sense of the preceeding remarks, we get the ECSFEXT  $S_n = (T_n(1), \dots, T_n(9))$ . We have

**Theorem 3.** Let  $k = [n^\alpha], \ell = [n^\beta], \frac{1}{2} < \beta < \alpha < 1, 0 < \delta < \frac{1}{2}$  such that  $\delta + \beta - 1 > 0, 2v = \min(1 - \alpha, \delta + \beta - 1)$ . Suppose that  $\pi_{7,9}(S_n) \rightarrow_p A \in \pi_{7,9}(\mathcal{S})$  as  $n \rightarrow +\infty$  with  $y_0 = +\infty$ . If further  $T_n(8, \beta/2) \rightarrow_p 0$  as  $n \rightarrow +\infty$ , then

- (i)  $F \in D(\Lambda)$  whenever  $c = 1$  and  $d = f = 0$
- (ii)  $F \in D(\phi_\gamma)$  whenever  $c = 1$  and  $d = f = \gamma^{-1}, \gamma > 0$ .

Now suppose that  $\pi_{7,9}(S_n) \rightarrow_p A \in \pi_{7,9}(\mathcal{S})$  as  $n \rightarrow +\infty$  with  $y_0 < +\infty$ . Then

- (iii) If  $c = 1, d = f = 0$  and  $T_n(8, \beta/2) \rightarrow_p 0$  as  $n \rightarrow +\infty$ , then  $F \in D(\Lambda)$ .
- (iv) If  $1 \leq c < \sqrt{2}, d = f = 0$  and  $T_n(9) \rightarrow_p 0$ , then  $F \in D(\psi_\gamma)$ , where  $\gamma = -2 + c^2 / (c^2 - 1)$ .

**Remark 3.** We clearly get here an ECSFEXT of nine statistics. The only concern is that the number is relatively high, since we need only one statistic for the Frechet domain. The main difficulty concerns de Gumbul subdomain. The idea behind the result of Mason is that the limit of the first asymptotic moment  $R(x, G)$  to a positive number is equivalent to  $F$  belongs to  $D(\phi)$ . For  $F \in D(\psi_\gamma)$ ,  $R(x, G)$  goes to zero as  $(x_0(G) - G^{-1}(1 - k/n))/(\gamma + 1)$ . But for  $F \in D(\Lambda)$ ,  $R(x, G)$  has as many as possible ways to tend to zero. This explains why the characterization of df's in  $D(\Lambda)$  requires a considerable number of statistics.

**Remark 4.** Diop and Lo (1994) [13] claimed an ECSFEXT of two statistics. They introduced the continuous generalized Hill's estimator

$$S_n(\tau) = k^{-\tau} \sum_{i=1}^k i^k (\log X_n - i + 1, n - \log X_{n-i,n})$$

where  $\tau > 0$  and  $k$  satisfies the usual condition and further thoroughly studied it in [15] and [14]. They indeed claimed that any couple of statistics  $(S_n(\tau), S_n(\rho))$ , for  $\tau \neq \rho$ , empirically characterizes the whole extremal domain of attraction. Further they acknowledged that their proof is wrong. However, any couple  $(S_n(\tau), S_n(\rho))$ ,  $\tau \neq \rho$ , is indeed an ESFEXT.

#### 4. PROOFS OF THE THEOREMS

Introduce the two first asymptotic moments

$$(4.1) \quad R(x, z, F) = (1 - F(x))^{-1} \int_x^z (1 - F(t)) dt,$$

$x < z \leq x_0(F)$  with  $R(x, x_0, F) \equiv R(x, F)$  and

$$(4.2) \quad W(x, z, F) = (1 - F(x))^{-1} \int_x^z \int_y^z (1 - F(t)) dt dy,$$

$x < z \leq x_0(F)$  with  $W(x, x_0, F) \equiv W(x, F)$ . Let

$$F^{-1}(u) = \inf \{x, F(x) \geq u\},$$

$0 < u \leq 1$ ,  $F^{-1}(0) = F^{-1}(0_+)$ , the generalized inverse function of  $F$  and let also  $F(1) = 1$ .

From now on,  $R(x, \cdot)$  and  $W(x, \cdot)$  are used only for  $G(x) = F(e^x)$ . The proofs are based on the technical tools in Section 6. We first say that **Fact 1** in Section 6 means

$$(4.3) \quad \{Y_{j,n}, 1 \leq j \leq n, n \geq 1\} =_d \{G^{-1}(U_{n-j+1}), 1 \leq j \leq n, n \geq 1\}$$

where  $=_d$  stands for equality in distribution.

#### Proof of Theorem 2.

Let  $F \in \Gamma$ ,  $u_n = 1 - G(Y_{n-k,n})$ ,  $v_n = 1 - G(Y_{n-\ell,n})$ ,  $k = [n^\alpha]$ ,  $\ell = [n^\beta]$ ,  $\frac{1}{2} < \beta < \alpha < 1$ . First, we have to prove that

$$(4.4) \quad 0 < v_n < u_n \rightarrow 0, a.s., \text{ and } v_n/u_n \rightarrow 0,$$

a.s. as  $n \rightarrow +\infty$ . By **Facts 1, 2 and 3** in Section 6, we have

$$(4.5) \quad \begin{aligned} v_n/u_n &= \frac{1 - G_n(Y_{n-\ell,n}) + 0(n^{-\delta})}{1 - G_n(Y_{n-k,n}) + 0(n^{-\delta})} = \frac{U_{\ell+1,n} + 0(n^{-\delta})}{U_{k+1,n} + 0(n^{-\delta})} = \\ &= (\ell/k) \frac{1 + 0(n^{-\beta-\delta+1})}{1 + 0(n^{-\alpha-\delta+1})} \rightarrow 0, a.s. \end{aligned}$$

as  $n \rightarrow +\infty$  whenever  $0 < \delta < \frac{1}{2}$ ,  $\beta + \delta - 1 > 0$ .

The proof of Theorem 2 will follow from the partial proofs of Statements (S1), (S2), etc.

(S1) :  $T_n(2, k, \ell) = R(x_n)(1 + o(1))$ , a.s., as  $n \geq 1$ ,

where  $x_n = Y_{n-k,n}$  and  $z_n = Y_{n-\ell,n}$ ,  $n \geq 1$ .

**Proof of (S1).** It is easy to check that

$$(4.6) \quad T_n(2, k, \ell) = nk^{-1} \int_{x_n}^{z_n} (1 - G_n(t)) dt.$$

By **Fact 2** in Section 6, we have for all  $\delta$ ,  $0 < \delta < \frac{1}{2}$ ,

$$(4.7) \quad T_n(2, k, \ell) = R(x_n, z_n) (1 + o(n^{-\alpha-\delta+1}(z_n - x_n) / R(x_n, z_n))), \text{ a.s.,}$$

as  $n \rightarrow +\infty$ . We choose  $\delta$  so that  $\mu = \beta + \delta - 1 > 0$ . By Lemmas 7 and 8 of in Section 6, Statements (4.3) and (4.4), we have

$$(4.8) \quad T(x_n, z_n)/R(x_n) \rightarrow 1, \text{ a.s., as } n \rightarrow +\infty$$

and then

$$(4.9) \quad T_n(2, k, \ell) = R(x_n)(1 + o(n^{-\mu}(z_n - x_n) / R(x_n))), \text{ a.s., as } n \rightarrow +\infty.$$

Now **either**  $G \in D(\phi_\gamma)$ ,  $\gamma > 0$  and we apply Formula 2.6.4 of De Haan (1970) ([6]):

$$R(x_n)/(y_0 - x_n) \rightarrow q = (1 - K)K^{-1}, \text{ a.s.,}$$

as  $n \rightarrow +\infty$  with  $K = (1 - 1/(\gamma + 2))$  and  $\frac{1}{2} < 1$ , to get

$$(4.10) \quad 0 \leq n^{-\mu}(x_n, z_n)/R(x_n) \leq \{2c^{-1}(x_n, z_n)/(y_0 - x_n)\} n^{-\mu} \leq \frac{2}{q} n^{-\mu} \rightarrow 0, ,$$

a.s., as  $n \rightarrow +\infty$ .

**Or**  $F \in D(\Lambda) \cup D(\phi_\gamma)$  for some  $\gamma > 0$ . By Lemma 4 in Section 6,  $u^{1/\gamma}F^{-1}(1 - u)$  is SVZ when  $F \in D(\phi_\gamma)$ . It may also be shown from (6.3) that  $F^{-1}(1 - u)$  is SVZ when  $F \in D(\Lambda)$ . Now, using (4.3) and the Karamata representation for  $F^{-1}(u)$ , we get in both cases for  $0 < \varepsilon < \gamma$ ,

$$(4.11) \quad \frac{1}{2}(u_n/v_n)^{\lambda-\varepsilon} \leq F^{-1}(X_{n-\ell,n})/F^{-1}(X_{n-k,n}) = F^{-1}(X_{n-k,n})$$

$$= F^{-1}(1 - U_{\ell+1,n})/F^{-1}(1 - U_{k+1,n}) \leq 2(u_n/v_n)\lambda + \varepsilon,$$

a.s., as  $n \rightarrow +\infty$ , where  $\lambda = 1/\gamma$  when  $F \in D(\phi_\gamma)$  or  $\lambda = 0$  when  $F \in D(\Lambda)$ . Since  $G^{-1}(1 - u) = \log F^{-1}(1 - u)$  and since  $u_n/v_n \sim n^{\alpha-\beta}$ , a.s. as  $n \rightarrow +\infty$ , it follows that

$$(4.12) \quad z_n - x_n = o(\log n), \text{ a.s. } n \rightarrow +\infty,$$

and thus



$$(4.13) \quad 0 \leq (z_n - x_n) / (n^\mu R(x_n)) = 0 \left( \frac{\log n}{n^{\mu/2}} \times \frac{1}{n^{\mu/2} R(x_n)} \right), \text{ a.s. as } n \rightarrow +\infty$$

By Lemma 6 in Section 6,  $R(G^{-1}(1-u))$  is SVZ, and since  $u_n \sim (k/n)$ , a.s.  $n \rightarrow +\infty$ ,  $R(x_n) \sim R(G^{-1}(1-k/n))$ , a.s., as  $n \rightarrow +\infty$ . Hence, by Lemma 4 in Section 6,

$$(4.14) \quad n^{\mu/2} R(x_n) \sim n^{\mu/2} R(G^{-1}(1-k/n)) \rightarrow +\infty, \quad \text{a.s., as } n \rightarrow +\infty.$$

(4.7), (4.9), (4.10), (4.13) and (4.14) together prove **(S1)**.

**(S2) :**

$A_n(1, k, \ell) = (T_n(2, k, \ell)/T_n(1, k, \ell))^2 = W(x_n)(1 + o(1)) = K \times R(x_n)^2(1 + o(1))$ , a.s., as  $n \rightarrow +\infty$ , where  $K = 1$  if  $F \in D(\Lambda) \cup D(\phi)$ , and  $K = 1 - 1/(\gamma + 2)$  if  $F \in D(\psi_\gamma)$ ,  $\gamma > 0$ .

**Proof of (S2).** We check that

$$A_n(1, k, \ell) = nk^{-1} \int_{x_n}^{z_n} \int_y^{z_n} 1 - G_n(t) dt dy.$$

By **Fact 2** in Section 6,

$$(4.15) \quad A_n(1, k, \ell) = W(x_n, z_n)(1 + 0(n^{-\mu}(z_n - x_n)^2 W(x_n, z_n))),$$

a.s., as  $n \rightarrow +\infty$ . By Lemma 7 in Section 6, and by Statements (4.3) and (4.4),

$$(4.16) \quad W(x_n)/R(x_n) \rightarrow 1,$$

a.s. as  $n \rightarrow +\infty$ . Hence, Lemma 2 in Section 6 yields

$$(4.17) \quad W(x_n)/R(x_n)^2 \rightarrow K, \text{ a.s., as } n \rightarrow +\infty$$

It follows from (4.15), (4.16) and (4.17) that

$$(4.18) \quad A_n(1, k, \ell) = W(x_n)(1 + 0(n^{-\mu}(z_n - x_n)^2 R(x_n)^{-2})), \text{ a.s., as } n \rightarrow +\infty$$

But the calculations that led to (4.10) and (4.13) showed that for all  $\rho > 0$ ,  $0 < \zeta \leq \xi$ ,

$$(4.19) \quad (x_n, z_n)^{\zeta n - \rho} R(x_n)^{-\xi} \rightarrow 0 \text{ a.s., as } n \rightarrow +\infty$$

whenever  $F \in \Gamma$ . Thus (4.17) and (4.18) ensure **(S2)**.

**(S3) :**  $T_n(1, k, \ell) \rightarrow K^{-1} K \sqrt{K}$ , a.s., as  $n \rightarrow +\infty$ .

**Proof of (S3).** (S1) and (S2) prove (S3).

(S4) :  $T_n(3, k, \ell, v) \rightarrow 0$ , a.s., as  $n \rightarrow +\infty$ .

**Proof of (S4).** (S1) implies that  $T_n(3, k, \ell, v) \sim n^{-v}(x_n, z_n)R(x_n)^{-1}$ , a.s., as  $n \rightarrow +\infty$ . Thus (4.19) completes the proof of (S4).

(S5) :  $T_n(4) \uparrow y_o$ , a.s., as  $n \rightarrow +\infty$ .

**Proof of (S5).** This fact is obvious.

(S6). We have

$$T_n(2, \ell, 1) \rightarrow \begin{cases} 1/\gamma & \text{if } F \in D(\phi_\gamma) \\ 0 & \text{if } F \in D(\psi) \\ 0 & \text{if } F \in D(\Lambda). \end{cases}$$

**Proof of (S6).** If  $F \in D(\phi_\gamma)$ ,  $T_n(2, \ell, 1) \rightarrow 1/\gamma$ , a.s. as  $n \rightarrow +\infty$  by Theorem 2 of Mason (1982) ([24]). For  $F \in D(\Lambda) \cup D(\psi)$ , use (4.6) and get, for  $0 < \mu < \frac{1}{2}$ ,

$$(4.20) \quad T_n(2, \ell, 1) \leq R(z_n) + 0(n^{-\mu}(Y_{n,n} - Y_{n-\ell,n}))$$

$$\leq R(z_n) + 0(n^{-\mu}\alpha_n),$$

a.s. as  $n \rightarrow +\infty$ . But  $\alpha_n \rightarrow 0$  as  $n \rightarrow +\infty$  when  $F \in D(\psi)$  since  $Y_{n,n} \uparrow y_0$  and  $Y_{n-\ell,n} \uparrow y_0 < +\infty$ . If  $F \in D(\Lambda)$ , it may be showed as in (4.12) that  $\alpha_n = 0_p(\log n)$  as  $n \rightarrow +\infty$ , that is

$$(4.21) \quad \lim_{\rho \uparrow +\infty} \mathbb{P}(\alpha_n > \rho \log n) = 0.$$

Hence in both cases,  $T_n(2, \ell, 1) \rightarrow_{\mathbb{P}} 0$ , since  $R(x_n) \rightarrow_{\mathbb{P}} 0$ , as  $n \rightarrow +\infty$ , by Lemma 1. The proof of (S6) is now complete.

(S7).  $T_n(6) \xrightarrow{p} 0$  as  $n \rightarrow +\infty$ .

**Proof of (S7).** We use the device of Fact 5 in (4.6) by considering the integral as an improper one with respect to the upper bound. Remarking that  $(\ell/n) \sim (1 - G(Y_{n-\ell,n}))$ , we get

$$(4.22) \quad T_n(6) \leq 2Z_n(1)R(z_n)/(z_n - x_n)$$

where  $Z_n(1) = \sup_{U_{1,n} \leq s \leq 1} |U_n(s)/s|$ . This together with Fact 4 and Lemma 8 in Section 6 ensures (S7).

(S8)  $T_n(7) \xrightarrow{p} 0$ , as  $n \rightarrow +\infty$ ,

**Proof of (S8).** As for  $T_n(6)$ , we have

$$(4.23) \quad T_n(7) \leq 2Z_n(1)W(z_n)/(z_n - x_n)^2 \xrightarrow{p} 0 \text{ as } n \rightarrow +\infty,$$

by the very same arguments. Thus  $T_n(7) \rightarrow_{\mathbb{P}} 0$ , as  $n \rightarrow +\infty$ , is proved.

**(S9).** If  $F \in D(\Lambda) \cup D(\phi)$ ,  $T_n(8) \rightarrow 0$ , a.s., as  $n \rightarrow +\infty$ .

**Proof of (S9).** We recall that  $T_n(8) = n^{-v}(z_n - x_n)^{-1}$ . By the DDHM's representation (cf. Lemma 4 in Section 6 and by (4.3)), we have for all  $\lambda > 1$ ,

$$(4.24) \quad z_n - x_n \geq G^{-1}(1 - \lambda \varepsilon_n) = s(\varepsilon_n) + \int_{\varepsilon_n}^{\lambda \varepsilon_n} s(t) t^{-1} dt,$$

for large  $n$ , where  $\varepsilon_n = U_{\ell+1,n}$ . Now, the properties of SVZ functions easily yield for any fixed  $\varepsilon$ ,  $0 < \varepsilon < \frac{1}{2}$ ,  $-\varepsilon + (1 - \varepsilon) \log \lambda \geq 1$ ,

$$(4.25) \quad z_n - x_n \geq (1 - \varepsilon) s(\ell/n) \{-\varepsilon + (1 - \varepsilon) \log \lambda\} \geq \frac{1}{2} s(\ell/n), \text{ a.s.},$$

as  $n \rightarrow +\infty$ . Thus, by Lemma 4 in Section 6,

$$(4.26) \quad n^\nu(x - y) \geq \frac{1}{2} n s(\ell/n) \rightarrow +\infty,$$

a.s. as  $n \rightarrow +\infty$ . The proof of (S10) is complete.

**(S10).** If  $F \in D(\psi)$ , then  $T_n(9) \rightarrow 0$ , a.s., as  $n \rightarrow +\infty$ .

**Proof of (S10).** It is already obtained in (4.25).

We now sum up our partial proofs to get Theorem 2 :

- (i) (S3) gives the two possible limits of  $T_n(3, k, \ell, v)$
- (ii) (S1) and Lemma 1 in Section 6 give the two possible limits of  $T_n(2, k, \ell)$ .
- (iii) (S4) gives the unique limit of  $T_n(3, k, \ell, v)$ .
- (iv) (S5) gives the limits of  $T_n(4)$
- (v) (S6) gives the two possible limits of  $T_n(2, \ell, 1)$ .
- (vi) (S7) gives the limit of  $T_n(6)$
- (vii) (S8) gives the limit of  $T_n(7)$ .

These points ensure Parts (i) and (ii) of Theorem 2. As to the part (iii), it is proved by (S9) and (S10).

**4.1. Proof of Theorem 3.** First, use **Fact 2** in Section 6 as in (4.6) and get

$$(4.27) \quad R(x_n, z_n) = T_n(2, k, \ell) = T_n(2, k, \ell)(1 + 0_p(n^{-2v}(z_n - x_n) / T_n(2, k, \ell))).$$

$$(4.28) \quad = T_n(2, k, \ell)(1 + 0_p(T_n(3, k, \ell, 2v)))$$

But  $T_n(3, k, \ell, 2v) \xrightarrow{p} 0$  and then as  $n \rightarrow +\infty$ .

$$(4.29) \quad R(x_n, z_n) = T_n(2, k, \ell)(1 + 0_p(1)),$$

Secondly,

$$(4.30) \quad W(x_n, z_n) = A_n(1, k, \ell)(1 + 0_p(n^{-2v}(z_n - x_n) / A_n(1, k, \ell))).$$

as  $n \rightarrow +\infty$ . Since  $T_n(1, k, \ell) \xrightarrow{p} K^{-1}$ ,  $c^2 = K^{-1}$ ,  $c^2 = K^{-1}$ ,  $A_n(1, k, \ell) = K^2 + 0_p(1)T_n(2, k, \ell)^2$  as  $n \rightarrow +\infty$ . Thus

$$(4.31) \quad W(x_n, z_n) = K T_n(2, k, \ell)^2(1 + 0_p(T_n(3, k, \ell, v)^2)), \text{ as } n \rightarrow +\infty$$

Since  $T_n(3, k, \ell, v) \xrightarrow{p} 0$ , as  $n \rightarrow +\infty$ , one has

$$(4.32) \quad W(x_n, z_n) = K T_n(2, k, \ell)^2(1 + 0_p(1)), \text{ as } n \rightarrow +\infty.$$

Formulas (4.29) and (4.32) together imply

$$(4.33) \quad W(x_n, z_n) / R(x_n, z_n)^2 \xrightarrow{p} K, \text{ as } n \rightarrow +\infty.$$

We now want to drop  $z_n$  in (4.33). It suffices to check whether the conditions of Lemma 7 are satisfied. For that, we use the device of Fact 5 for  $T_n(2, k, \ell)$  see (4.22) to get

$$(4.34) \quad T_n(2, k, \ell) \geq Z_n(2)(R(z_n) - (n/\ell)(1 - G(Y_{n,n}))R(Y_{n,n})),$$

where  $Z_n(2) = \inf_{U_{1,n} \leq s \leq 1} |U_n(s) / s|$ . But for all d.f.  $G$ ,  $(G^{-1}(u)) \geq u$ . Thus, by applying (4.3), one has

$$(4.35) \quad -\frac{n}{\ell}(1 - G(G^{-1}(1 - U_{1,n}))) \geq nU_{1,n} / \ell = -0_p^+(1) / \ell, \text{ as } n \rightarrow +\infty.$$

where  $X = 0_p^+(1)$  means that  $P(X < 0) = 0$  and  $X = 0_p(1)$ . By Statement (4.63) below,  $R(Y_{n,n}) = 0_p^+(1)$  as  $n \rightarrow +\infty$ . Finally, we arrive at

$$(4.36) \quad T_n(6) \geq Z_n(2)(R(z_n) / (z_n - x_n) - n^{-\beta}(z_n - x_n)^{-1}0_p(1)), \text{ as } n \rightarrow +\infty$$

and

$$(4.37) \quad T_n(6) \geq Z_n(2)(R(z_n) / (z_n - x_n) - T_n(8, \beta)0_p(1)), \text{ as } n \rightarrow +\infty$$

But  $T_n(8, \beta/2) \xrightarrow{p} 0$  by assumption. Thus  $T_n(8, \beta) \xrightarrow{p} 0$  and by (4.37) and by Fact 4,

$$(4.38) \quad R(z_n) / (z_n - x_n) \xrightarrow{p} 0, \text{ as } n \rightarrow +\infty$$

We also have, if  $y_0 < +\infty$ ,

$$(4.39) \quad R(z_n) / (z_n - x_n) \leq (y_0 - z_n) / (z_n - x_n) = (-1 + T_n(9)^{-1})^{-1}$$

If  $T_n(9) \xrightarrow[p]{} 0$ , i.e.,  $T_n(9)^{-1} \rightarrow +\infty$ , then

$$(4.40) \quad R(z_n) / (z_n - x_n) \xrightarrow[p]{} 0, \text{ as } n \rightarrow +\infty.$$

This is the first condition of Lemma 7. For the second, we remark that

$$(4.41) \quad A_n(1, k, \ell) \geq Z_n(2) (W(z_n) - \ell^{-1} 0_p(1) W(Y_{n,n})) , \text{ as } n \rightarrow +\infty$$

Since  $W(Y_{n,n}) = 0_p^+(1)$  by Lemma 1 and Statement (4.68) below, we obtain

$$(4.42) \quad T_n(7) \geq Z_n(2) \{W(z_n) / (z_n - x_n)^2 - T_n(8, \beta)^2 0_p^+(1)\}.$$

By the same reasons that gave (4.38), we arrive at

$$(4.43) \quad W(z_n) / (z_n - x_n) \xrightarrow[p]{} 0, \text{ as } n \rightarrow +\infty,$$

whenever  $T_n(8, \beta) \xrightarrow[p]{} 0$ , as  $n \rightarrow +\infty$ . We also have, when  $y_0 < +\infty$ ,

$$(4.44) \quad W(z_n) / (z_n - x_n)^2 \leq (y_0 - z_n)^2 / (z_n - x_n)^2 = (-1 + T_n(9)^{-1})^{-2}.$$

Hence  $T_n(9) \xrightarrow[p]{} 0$ , as  $n \rightarrow +\infty$ , implies

$$(4.45) \quad W(z_n) / (z_n - x_n) \xrightarrow[p]{} 0, \text{ as } n \rightarrow +\infty.$$

We have proved that the conditions of Lemma 7 are satisfied via (4.38), (4.38), (4.40), (4.43) and (4.45). Thus (4.33) becomes

$$(4.46) \quad W(x_n)/R(x_n)^2 \rightarrow_p K, \text{ as } n \rightarrow +\infty, \text{ with } 1 \leq K < \sqrt{2}.$$

We now show how the preceeding may prove Theorem 2

If  $T_n \rightarrow_p A$  with  $d = 1/\gamma$ ,  $0 < \gamma < +\infty$ , thus by Mason (1982) ([24]),  $F \in D(\phi_\gamma)$  and all the other limits of  $T_n(i), i = 1, \dots, 8$  are justified by Theorem 1.

If  $T_n \rightarrow_p A$  with  $c = 1$  and  $T_n(8, \beta) \rightarrow_p 0$ , thus we get (4.46) with  $K = c^{-2} = 1$ .

If  $T_n \rightarrow_p A$  with  $y_0 < +\infty$  and  $1 \leq c < \sqrt{2}$  and  $T_n(+9) \rightarrow_p 0$ , thus (4.46) holds with  $K = c^{-2} = 1 - 1/(\gamma + 2)$ , for some  $\gamma$ ,  $0 < \gamma < +\infty$ .

It is clear now that Lemma 2 ensures from (ii) that  $F \in D(\Lambda)$  and from (iii) that  $F \in D(\psi_\gamma)$  if we show

$$(4.47) \quad \lim_{x \rightarrow y_0} W(x)/R(x)^2 = \lim_{n \rightarrow +\infty} W(x_n)/R(x_n)^2.$$

**Proof of (4.47).** Recall basic facts

$$(4.48) \quad n^{-\rho} \leq n^{-\rho} + n^{-1}(k+1)/n \leq n^{-\rho} + 2n \quad \text{for } k = [n^\alpha], \quad \rho = 1 - \alpha$$

By Fact 2, for  $0 < \delta < \frac{1}{2}, 0 < \delta < +\alpha - 1 < 1, 0 < \varepsilon < 0.01$ ,

$$(4.49) \quad n^{-\rho} - \varepsilon n^{-\delta} \leq U_{k+1,n} \leq n^{-\rho} + 2\varepsilon n^{-\delta},$$

a.s., as  $n \rightarrow +\infty$ .

Set  $a_n = n^{-\rho}$ ,  $n \geq 1$ . This sequence  $a_1 = 1 > \dots > a_j > a_{j+1} > \dots$  makes a partition of  $[0, 1]$ . For  $x \uparrow y_0, u = 1 - G(x) \downarrow 0$ , there exists at each step of this limit an integer  $n$  such that  $a_{n+2} \leq u \leq a_{n+1} < a_n$ . Let  $m = m(n) = n - [n^\tau]$ ,  $\tau = 2 - \alpha - \delta$ . Remark that  $\frac{1}{2} < \tau < 1$ . One has  $a_m - a_n = -[n^\tau] f'(\zeta_n)$  with  $f(x) = x^{-\rho}$  and  $m \leq \zeta_n \leq n$ . Thus for large values of  $n$ ,

$$(4.50) \quad a_m - a_n \geq (1 - \varepsilon)n^{-\delta}.$$

By the preceeding facts, for large  $n$ ,

$$(4.51) \quad U_{k(m)+1,m} \geq a_m - \varepsilon(1 + \varepsilon)n^{-\delta}.$$

Since  $m \rightarrow +\infty$  as  $n \rightarrow +\infty$ , it follows that

$$(4.52) \quad a_{n+2} \leq u \leq a_{n+1} < a_n \leq a_m - (1 - \varepsilon)n^{-\delta} \leq a_m - \varepsilon(1 + \varepsilon)n^{-\delta} \leq U_{k(m)+1,m}$$

and

$$(4.53) \quad u/U_{k(m)+1,m} \rightarrow_p \text{as } n \rightarrow +\infty, u \rightarrow 0$$

Put

$$M(x) = \int_x^{y_0} 1 - G(t)dt \quad \text{and} \quad m(x) = \int_x^{y_0} \int_y^{y_0} (1 - G(t)) dy dt.$$

Using the inequality  $G^{-1}(G(x)) \leq x$  for all  $x$  and for all  $df$   $G$  and noticing that both  $M(\cdot)$  and  $m(\cdot)$  are nonincreasing, we obtain

$$(4.54) \quad 0 \leq M(G^{-1}(1 - U_{k+1,m})) - M(G^{-1}(1 - u)) \leq M(x_m) - M(x) = \int_{x_m}^x 1 - G(t)dt.$$

Because of Lemma 5, either  $x = G^{-1}(1 - u)$  or  $x$  lies on the constancy interval of  $G$ ,  $[G^{-1}(1 - u), G^{-1}(1 - u+)] = ]y, z]$ . Hence

$$(4.55) \quad \begin{aligned} 0 \leq M(x_m) - M(x) &\leq \int_{x_m}^y 1 - G(t)dt + (z - y)u \\ &\leq \{G^{-1}(1 - u) - x_m\} + (z - y)u. \end{aligned}$$

One may quickly check that for large values of  $n$ ,

$$(4.56) \quad \begin{cases} 0 \leq 1 - G(x_m) \sim U_{k(m)+1,m} \sim u, & (\text{ see Statement (4.4)}) \\ 0 \leq G^{-1}(1-u) - x_m \leq z_m - x_m, \\ 0 \leq z - y = G^{-1}(1-u) - G^{-1}(1-u) \leq z_m - x_m \end{cases}$$

Hence,

$$(4.57) \quad \begin{aligned} & 0 \leq 1 - (1 + 0_p(1))R(x)/R(x_m) \\ & \leq (z_m - x_m)/(m^{\alpha-1}R(x_m)) + 2(z_m - x_m)/R(x_m), \end{aligned}$$

as  $n \rightarrow +\infty$ . Using (6.14), we finally get

$$(4.58) \quad \begin{aligned} & 0 \leq 1 - (1 + 0_p(1))R(x)/R(x_m) \\ & \leq (z_m - x_m)/(m^{\alpha-1}R(x_m)) + 2 \left(1 + \frac{z_m - x_m}{R(z_m)}\right)^{-1} \frac{z_m - x_m}{m^{\alpha-\beta}R(z_m)} \end{aligned}$$

$$(4.59) \quad \leq \frac{z_m - x_m}{m^{2v}R(z_m)} + m^{\beta-\alpha} \sup_{x \geq 0} |x(1+x)| \leq \frac{z_m - x_m}{m^{2v}R(z_m)} + m^{\beta-\alpha},$$

when  $n$  is large enough. Now from (4.6)

$$(4.60) \quad m^{2v}T_n(2, k(m), \ell(m))/(z_m - x_m) = T_m(3, k, \ell, v)^{-1} \leq Z_m(1) \frac{m^{2v}R(x_m - z_m)}{z_m - x_m},$$

where  $Z_m(1)$  is defined in (4.22). Since  $T_m(3, v) \xrightarrow[p]{p} 0$  as  $p \rightarrow +\infty$ , by assumption, and since  $Z_m = 0_p(1)$ , we get

$$(4.61) \quad (z_m - x_m)/(m^{2v}R(x_m, z_m)) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

This and (4.60) together imply

$$(4.62) \quad \lim_{x \rightarrow y_0} R(x)/R(x_m) = 1 \quad \text{in probability.}$$

It follows that

$$(4.63) \quad \lim_{x \rightarrow y_0} R(x) = \lim_{n \rightarrow +\infty} R(x_n) \quad \text{in probability}$$

By the very same arguments, one gets

$$\begin{aligned} & 0 \leq 1 - (1 + 0_p(1))W(x)/W(x_m) \\ & \leq m^{-(1-\alpha)} \{G^{-1}(1-u) - x_m\}^2 / W(x_m) + 3(y - z)^2 / W(x_m) \\ & \leq (z_m - x_m)^2 / (m^{2v}W(x_m)) + 3 \frac{(x_m - z_m)^2}{W(z_m)} \frac{W(z_m)}{W(x_m)}, \text{ as } n \rightarrow +\infty. \end{aligned}$$

Now, by using (6.18), we arrive at

$$(4.64) \quad 0 \leq 1 - (1 + 0_p(1))W(x) / W(x_m) \leq (z_m - x_m)^2 / (m^{2v}W(x_m)) + 3m^{(\alpha-\beta)} \sup_{x \geq 0} |x(1+x)|,$$

as  $n \rightarrow +\infty$ . Taking (4.47) into account gives for large values of  $n$ ,

$$(4.65) \quad 0 \leq 1 - (1 + 0_p(1))W(x) / W(x_m)$$

$$(4.66) \quad \leq 2 \{(z_m - x_m)/(m^{2v}R(x_m))\}^2 + 3m^{-(\alpha-\beta)},$$

which in turn implies

$$(4.67) \quad \lim_{x \rightarrow y_0} W(x) / W(x_m) = 1 \text{ in probability.}$$

It follows that

$$(4.68) \quad \lim_{x \rightarrow y_0} W(x) = \lim_{n \rightarrow +\infty} W(x_n) \text{ in probability.}$$

Now Formulas (4.62) and (4.68) together give (4.47) which, combined with Lemma 4 proves Theorem 3.

## 5. CONCLUDING COMMENTS

**5.1. Conjecture.** We conjecture that the couple  $(A_n, T_n)$  should suffice to characterize the whole extremal domain, in particular that of the Gumbel subdomain, following the de Haan's functional characterization of  $D(\Lambda)$  and  $D(\psi)$  (Theorems 2.5.6 and 2.6.1 in [6] as reminded in Lemma 2). As well, it must be expected, unless a counterexample is given, that the couple of Diop and Lo statistics, given in Remark 4, would also be an ECSFEXT.

**5.2. Technical improvements.** The restriction  $\beta > \frac{1}{2}$  is required just for (4.4). It is easily showed that when  $F \in \Gamma$ , one has  $\lim_{u \rightarrow 0} (1 - G(G^{-1}(1 - u)))/u = 1$ . This remarks remove the conditions  $\beta > \frac{1}{2}$  in Theorem 1. For the weak limit, it will be shown in the coming paper that Theorem 1 holds for all sequences  $k$  and  $\ell$  whenever  $k/n \rightarrow 0$ ,  $\ell/n \rightarrow 0$ ,  $\ell/k^{\frac{1}{2}-\eta} \rightarrow 0$ , as  $n \rightarrow +\infty$ , for some  $\eta$ ,  $0 \leq \eta < \frac{1}{2}$  ( $\eta = 0$  for  $F \in D(\psi) \cup D(\phi)$ )

**5.3. Multivariate Gaussian Law of the ECSFEXT.** The existence of family of statistics characterizing some class of distribution must yield statistical tests. The first step to this is the determination of the limit laws of the ECSFEXT. This is done in a coming paper.

## 6. TECHNICALS LEMMASS

We invite the reader to remind the definitions of the two first asymptotic moments of a distribution function in 4.1 and 4.2. We have the following properties.



**Lemma 1.** *For any  $\gamma$ ,  $0 < \gamma < +\infty$ ,*

- (i)  *$F \in D(\Lambda)$  iff  $G \in D(\Lambda)$  and  $R(x, G) \rightarrow 0$  as  $x \rightarrow x_0(G) = y_0$ ;*
- (ii)  *$F \in D(\Phi_\gamma)$  iff  $G \in D(\Lambda)$  and  $R(x, G) \rightarrow 1/\gamma$  as  $x \rightarrow x_0(G) = y_0$ ;*
- (iii)  *$F \in D(\psi_\gamma)$  iff  $G \in D(\psi_\gamma)$  and then  $R(x, G) \rightarrow 0$  as  $x \rightarrow y_0 < +\infty$ ;*
- (iv)  *$F \in D(\psi_\gamma)$  iff  $F(x_0 - 1/\cdot) \in D(\psi_\gamma)$ .*

*Proof.* See Lemmas 9 and 10 in Lô (1986) for (i) and Lemma 1 in Mason (1982) for (ii). Point (iii) is proved similarly to (i) and (ii). (iv) is Part (ii) of Theorem A. of de Haan (1970).  $\square$

**Lemma 2.** *(de Haan [6], Theorems 2.5.6 and 2.6.1). We have*

- (i)  *$F \in D(\Lambda)$  iff  $W(x, F)$  and  $W(x, F)/R(x, F)^2 \rightarrow 1$ , as  $x \rightarrow x_0$ .*
- (ii)  *$F \in D(\psi_\gamma)$  iff  $x_0(F) + \infty$  and  $W(x, F)/R(x, F)^2 \rightarrow (1 - 1/(2 + \gamma))$ , as  $x \rightarrow x_0$ .*

Functions  $s(u)$ ,  $0 < u < 1$ , such that for all  $\lambda > 0$ ,

$$\lim_{u \rightarrow 0} s(\lambda u)/s(u) = 1,$$

are called Slowly Varying functions at Zero (SVZ) and are greatly involved in our proofs. We recall here some of their properties before we state some basic results of  $df$ 's lying in  $\Gamma$ .

**Lemma 3.** *Let  $s(u), 0 < u < 1$ , be SVZ. Then,*

(i) *It admits the Kamarata's representation (KARARE) :*

$$(6.1) \quad s(u) = c(u) \exp \left( \int_u^1 b(t) t^{-1} dt \right), \quad 0 < u < 1$$

where  $c(u) \rightarrow c, 0 < c < +\infty, b(u) \rightarrow 0$ , as  $u \rightarrow 0$ .

(ii) *For any  $\delta > 0, u^{-\delta} s(au) \rightarrow 0$ , as  $au \rightarrow 0$  and  $a \times u \rightarrow 0$ .*

*Proof.* (6.1) in (i) is well-known. See Lemma 12 of L $\hat{o}$  (1986a) ([19]) for (ii). Anyway it is easily derived from (6.1).  $\square$

**Lemma 4.** *We have*

- (i)  *$F \in D(\Phi_\gamma)$  iff  $u^{1/\gamma} F^{-1}(1-u)$  is SVZ*
- (ii)  *$F \in D(\psi_\gamma)$  iff  $u^{1/\gamma} (x_0(F) - F^{-1}(1-u))$  is SVZ*
- (iii)  *$F \in D(\Lambda)$  iff there exists a SVZ function  $s(u), 0 \leq u \leq 1$  and a constant  $b$  such that*

$$(6.2) \quad F^{-1}(1-u) = b - s(u) + \int_u^1 s(t) t^{-1} dt, \quad 0 < u < 1$$

When 6.2) holds, one may take

$$(6.3) \quad s(u) = u^{-1} \int_{1-u}^1 (1-s) dF^{-1}(s) =: r(u, F).$$

*Proof.* See Theorem 2.4.1 of De Haan (1970) for (i). (ii) is easily derived from (i) and Part (iii) of theorem A. The representation 6.2 is due to De Haan (1970) [6]. The writing of  $s(u)$  as (6.3) is due to Deheuvel-Haeusler-Mason (1988) [?].  $\square$

**Lemma 5.** *Let  $G$  be any distribution function. Then*

- (i) *for all  $0 < u < 1, G(G^{-1}(u)) = u$  or  $u$  lies on a constancy interval of  $G^{-1}$ .*
- (ii) *for all  $-\infty < x < +\infty, G^{-1}(G(x)) = x$  or  $x$  lies on a constancy interval of  $G$ .*

*Proof.* Notice that the set of discontinuity points of  $G$ , say  $D$ , is countable. And for all  $x \in D, [G(x-), G(x)] =: [v_x, u_x]$  is a constancy interval of  $G^{-1}$  with

$$(6.4) \quad \forall u \in [v_x, u_x], G^{-1}(u) = x$$

Now let  $I = \bigcup_{x \in D} [v_x, u_x[$ . One has

$$(6.5) \quad \forall u \in I, \quad G(G^{-1}(u)) > u$$

One also has

$$(6.6) \quad \forall x \in D, \quad G(G^{-1}(u_x)) = u_x$$

It follows that the complementary  $J$  of  $I$  in  $(0, 1)$  is not empty. To finish, we have to show

$$(6.7) \quad (u \in J \text{ and } x = G^{-1}(u)) \Rightarrow (G(x) = u).$$

Suppose that for some  $u \in J, x = G^{-1}(u)$  and  $G(x) > u$ . Thus

**either** ,  $x$  is a continuity point and there exists a sequence  $x_n \uparrow x$  such that  $G(x_n) \uparrow G(x)$  as  $n \rightarrow +\infty$ . Hence for some  $\eta, n > \eta, x_n > u$  and  $G(x_n) > u$  so that  $G^{-1}(u) < x$ , which leads to a contradiction ;

**or**  $x$  is a discontinuity point and thus  $u$  lies on  $[v_x, u_x[$ , which is a constancy interval of  $G^{-1}$  and, by consequence,  $x \in I$ . This also leads to a contradiction. These two contradictions imply 6.7 which, combined with 6.5, prove Part (i).  $\square$

**Lemma 6.** *Let  $G \in D(\Lambda)$  then*

(i)  $(1 - G(G^{-1}(1 - u)))/u \rightarrow 1$  as  $u \rightarrow 0$

and

(ii)  $R(G^{-1}(1 - u))$  is SVZ.

*Proof.* **Proof of Part i.** **Either**  $(1 - G(G^{-1}(1 - u))) = 1 - u$  and thus

$$(6.8) \quad (1 - G(G^{-1}(1 - u)))u = 1;$$

**Or**, by Lemma 5,  $1 - u \in [G(x-), G(x)[$ , for some discontinuity point  $x$ , with  $1 - G(x) < u \leq 1 - G(x-)$  and hence

$$(6.9) \quad 1 - G(x) = (1 + c(x)) \exp \left( \int_{-\infty}^x \Phi(t)^{-1} dt \right), -\infty < x < x_0(G) = y_0,$$

where  $c(x) \rightarrow 0$ ,  $\Phi'(x)$  exists and  $\Phi'(x) \rightarrow 0$  as  $x \rightarrow y_0$ , yields

$$(6.10) \quad (1 - G(x)) / (1 - G(x-)) = (1 + c(x)) / (1 + c(x-)) \rightarrow 1 \text{ as } x \rightarrow y_0$$

This together with (6.8), and (6.9) prove Part i).

**Proof of Part ii.** From (6.2), one has (cf. Lemma 4 in L $\hat{o}$  (1989)),

$$(6.11) \quad \{G^{-1}(1 - \lambda u) - G^{-1}(1 - u)\} / s(u) \rightarrow -\log \lambda, \text{ as } u \rightarrow 0 ;$$

Thus

$$(6.12) \quad R(G^{-1}(1 - u)) \sim s(u) \text{ as } u \rightarrow 0,$$

which proves Part ii) since  $s(u)$  is SVZ.  $\square$

We now introduce two useful and important lemmas.

**Lemma 7.** . *Let  $F$  be any distribution function satisfying*

- (i)  $R(x)$  and  $W(x)$  are finite for  $x < x_0(F)$  ;
- (ii)  $(z - x)/R(z) \rightarrow +\infty$ , as  $x \rightarrow x_0$ ,  $x < z$  ;
- (iii)  $(z - x)^2/W(x) \rightarrow +\infty$ , as  $x \rightarrow x_0$ ,  $z \rightarrow x_0$ ,  $x < z$ ;

*Then,  $R(x, z) = R(x) \rightarrow 1$  and  $W(x, z)/W(x) \rightarrow 1$  as  $x \rightarrow x_0$ ,  $z \rightarrow x_0$ ,  $x < z$ .*

*Proof. Point (i).* We have, for  $x < z$ ,

$$(6.13) \quad R(x, z) = R(x) \left( 1 - \left\{ \int_z^{x_0} 1 - F(t) dt \right\} / \left\{ \int_z^{x_0} 1 - F(t) dt \right\} \right) =: R(x)(1 - E_0)$$

Since  $(1 - F(t))/(1 - F(z)) \geq 1$  when  $x \leq t \leq z$ , we get

$$(6.14) \quad 0 \leq E_1 = \left( 1 + \left\{ \int_x^{z_0} 1 - F(t) dt \right\} / \left\{ \int_z^{x_0} \int_x^{z_0} 1 - F(t) dt dy \right\} \right)^{-1}$$

**Point (ii).** Formula (6.14) and Assumption (ii) allow to conclude that

$$(6.15) \quad R(x, z) = R(x) \rightarrow 1 \text{ as } x \rightarrow x_0, \quad z \rightarrow x_0, \quad x < z.$$

**Point (iii).** We have

$$(6.16) \quad \begin{aligned} W(x, z) &= W(x) \left( 1 - \left( \left\{ \int_z^{x_0} \int_y^{x_0} 1 - F(t) dt dy \right\} / \left\{ \int_z^{x_0} \int_y^{x_0} 1 - F(t) dt dy \right\} \right) \right. \\ &\quad \left. - \left( \left\{ \int_x^z \int_z^{x_0} 1 - F(t) dt dy \right\} / \left\{ \int_z^{x_0} \int_y^{x_0} 1 - F(t) dt dy \right\} \right) \right) \\ &= W(x)(1 - E_1 - E_2) \end{aligned}$$

We approximate  $E_1$  and  $E_2$ . First, the inequality  $(1 - F(t)) \geq (1 - F(z))$  for  $x \leq t \leq z$  and Assumption (iii) yield

$$(6.17) \quad 0 \leq E_1 = (1 + (z - x)^2 / W(z))^{-1} \rightarrow 0 \text{ as } x \rightarrow x_0, \quad y \rightarrow x_0, \quad x < z.$$

Secondly,

$$(6.18) \quad \begin{aligned} 0 \leq E_2 &= (z - x) \left( \int_z^{x_0} 1 - F(t) dt / \int_z^{x_0} \int_x^{z_0} 1 - F(t) dt dy \right) \\ &= (z - x) \frac{R(z)}{W(z)} E_1 \leq \frac{R(z)}{z - x} \frac{(z - x)^2}{w(z)} (1 + (z - x)^2 / W(z))^{-1} \\ &\leq \sup_{x \in R^+} |(x(1 + x))^{-1}| \frac{R(z)}{z - x} \leq R(z) / (z - x) \rightarrow 0, \end{aligned}$$

as  $x \rightarrow x_0$ ,  $z \rightarrow x_0$ ,  $x < z$ , by Assumption (i). Statements 6.16, 6.17 and 6.18 yield

$$(6.19) \quad W(x, z) / W(x) \rightarrow 1 \text{ as } x \rightarrow x_0, \quad z \rightarrow x_0, \quad x < z.$$

Now Formulas 6.15 and 6.19 together prove Lemma 7.  $\square$

**Lemma 8.** Let  $F \in \Gamma$ , then for  $u = 1 - G(x)$ ,  $v = 1 - G(z)$ ,  $v/u \rightarrow 0$  as  $x \rightarrow y_0$ ,  $z \rightarrow y_0$ .

*Proof.* Let  $F \in \Gamma$ . Either  $G \in D(\Lambda)$  or  $G \in D(\psi_\gamma)$ ,  $\gamma > 0$ . It suffices to prove Part (i). Part (ii) will follow from Part (i) and Lemma 2. If  $G \in D(\Lambda)$ , Lemma 1 in L6 (1986a) implies Part (i).

If  $G \in D(\psi_\gamma)$ ,  $G(y_0 - 1/\cdot) \in D(\phi_\gamma)$  and thus  $(y_0 - y)^{-\gamma}(1 - G(y))$  is SV at infinity. KARARE yields

$$(6.20) \quad 1 - G(y) = c(y)(y_0 - y)^\gamma \exp\left(\int_{y_1}^{(y_0 - y)^{-1}} b(t)^{-1} dt\right), (y_0 - y)^{-1} > y_1,$$

where  $c(y) \rightarrow c$ ,  $0 < c < +\infty$ , as  $y \rightarrow y_0$  and  $b(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . for  $\eta$  such that

$$(6.21) \quad \sup_{t \geq \eta} b(t) \leq \varepsilon < \gamma, \quad y_0 - \eta^{-1} \leq y < y_0$$

Formula (6.20) implies

$$(6.22) \quad C_1 \cdot ((y_0 - x)/(y_0 - z))^{\gamma + \varepsilon} \leq (1 - G(x))/(1 - G(z)) = u/v \leq C_2 \cdot ((y_0 - x)/(y_0 - z))^{\gamma - \varepsilon}$$

where  $C_1$  and  $C_2$  are positive constants. The right inequality ensures that

$$(6.23) \quad (y_0 - x)/(y_0 - z) \rightarrow +\infty, \text{ as } u \rightarrow 0, \quad v/u \rightarrow 0.$$

By using now Formula 2.6.4 of De Haan [6], we get for some constant  $C$ ,

$$(z - x)/R(z) \sim C \quad (z - x)/(y_0 - z)$$

$$(6.24) \quad \text{Const. } (-1 + (y_0 - x)/(y_0 - z)) \rightarrow +\infty \\ \text{as } x \rightarrow y_0, z \rightarrow y_0, (1 - G(z))/(1 - G(x)) \rightarrow 0.$$

Part (i) is now proved. Part (ii) follows from Lemma 2.  $\square$

To finish with this section, we recall properties of empirical distribution functions (*edf*). The *edf* associated with  $Y_1, \dots, Y_n$  is defined by

$$(6.25) \quad G_n(x) = \# \{i, \quad 1 \leq i \leq n, \quad Y_i \leq x\} / n, \quad x \in R$$

Let  $U_n(s)$ ,  $0 \leq s \leq 1$ , be the *edf* associated with  $U_1, \dots, U_n$ , a.s.i.c. of a uniform *rv* on  $(0, 1)$ .

**Fact 1.** We may WLOG and do assume that

$$\{1 - G_n(x), \quad x \in R, \quad n \geq 1\} = \{U_n(1 - G(x)), \quad x \in R, \quad n \geq 1\}$$

**Fact2.** For all  $\mu$ ,  $0 < \mu \leq \frac{1}{2}$ ,

$$\limsup_{n \rightarrow +\infty} n^\mu \sup_{0 \leq s \leq 1} |U_n(s) - s| < \infty. \text{ a.s.}$$

**Fact 3** Let  $k = [n^\alpha]$ ,  $\frac{1}{2} < \alpha < 1$ . Then  $kU_{k,n}/n \rightarrow 1$ , a. s. as  $n \rightarrow +\infty$ .

**Fact 4.** We have

$$\lim_{\lambda \uparrow +\infty} \liminf_{n \uparrow +\infty} P(\lambda^{-1} \leq \inf_{U_{1,n} \leq s \leq 1} U_n(s)/s \leq \sup_{U_{1,n} \leq s \leq 1} U_n(s)/s \leq \lambda) = 1.$$

Fact 2 is derived from Theorem 4.5.2 and Formula 1.2.3 both in M. Csörgö-Révész (1981). Fact 3 is a consequence of Fact 2. Fact 4 is quoted in S. Csörgö and al.(1985) for the quantile process. A simple change of variable suffices to put it in the form of Fact 4.

We now introduce a general device which permits to overcome discontinuity problems.

**Fact 5.** For  $n$  fixed, there exists a sequence  $(t_p)_{p \geq 1}$  such that  $t_p \uparrow Y_{n,n}$  as  $p \uparrow +\infty$  and for all  $p \geq 1$ ,

$$\inf_{U_{1,n} \leq s \leq 1} U_n(s)/s \leq \inf_{0 \leq x \leq t_p} \frac{U_n(1 - G(x))}{1 - G(x)} \leq \sup_{0 \leq x \leq t_p} \frac{U_n(1 - G(x))}{1 - G(x)} \leq \sup_{U_{1,n} \leq s \leq 1} U_n(s)/s.$$

**Proof.** By using the representations of the constancy intervals of  $G^{-1}$  given in the proof of Lemma 5, we remark that :

(i) either  $G(Y_{n,n}) = G(G^{-1}(1 - U_{1,n})) = 1 - U_{1,n}$  and it suffices to put  $t_p = Y_{n,n}$  for all  $p \geq 1$ ;

(ii) or  $G^{-1}(1 - U_{1,n}) > 1 - U_{1,n}$  and, necessarily,  $1 - U_{1,n}$  lies in some constancy interval of  $G^{-1}, [v_j, u_j]$ . Putting  $t_p = Y_{n,n} - 1/p$  for  $p \geq 1$ , one has  $G(t_p) \leq v_j \leq 1 - U_{1,n}$  for all  $p \geq 1$ . Thus for  $0 \leq x \leq t_p$ ,  $1 - G(x) \geq U_{1,n}$ , for all  $p \geq 1$ . This completes the proof.

## 7. APPENDIX

**7.1. A counterexample about the de Haan-Resnick.** We show here that the De Haan-Resnick estimator for the index of a stable law defined by

$$(7.1) \quad C_n = \frac{Y_{n,n} - Y_{n-k,n}}{\log k}$$

does not characterize  $D(\phi_\gamma)$  as Hill's estimator does. To prove this, we begin to remark, as Mason (1982) ([24], (cf. its appendix) showed it, that the df  $G$  defined by

$$(7.2) \quad G^{-1}(1 - 2^{-m}) = m, \quad m = 0, 1, 2, \dots$$

and

$$G^{-1}(1-u) = m + (2^{-m} - u)(2^{m+1}), \text{ if } 2^{-m-1} < u < 2^{-m},$$

does not belong to  $D(\phi)$ . Thus by Lemma 1,  $F(\cdot) = G(\log(\cdot))$  does not belong to  $\Lambda$ . However, it is easy to check that

$$(7.3) \quad \frac{G^{-1}(1-s) - G^{-1}(1-bs)}{\log b} \rightarrow (\log 2)^{-1}, \text{ as } s \rightarrow 0, b \rightarrow +\infty \text{ and } bs \rightarrow 0$$

Letting  $b = U_{k,n}/U_{1,n}$ , we get, via representation (4.3),

$$(7.4) \quad C_n \rightarrow_p (\log 2)^{-1}, \text{ as } n \rightarrow +\infty$$

Thus the convergence of  $C_n$  to a positive and finite real number for all sequences  $k \rightarrow +\infty$  verifying  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ , does not imply that  $F$  belongs to  $D(\phi)$ .

**7.2. A useful identity that links Lo and Dekkers *et al.* estimators.** We prove here the following identity in the following

**Lemma 9.** *Let  $1 \leq k \leq n$  be integers and  $x_k, \dots, x_n$  ( $n-k+1$ ) real numbers. The we have*

$$(7.5) \quad \sum_{i=1}^k (x_{n-i+1} - x_{n-k})^2 = 2 \sum_{i=1}^k \sum_{j=1}^i j(1-\delta_{ij}/2)(x_{n-i+1} - x_{n-i})(x_{n-j+1} - x_{n-j});$$

where  $\delta_{ij} = 1$  if  $i = j$  and 0 elsewhere, is the Kronecker symbol.

*Proof.* We use these notations  $S_r = \sum_{1 \leq i \leq k} x_{n-j+1}^r$ ,  $r = 1, 2$ . We also these two formulas :

$$(7.6) \quad \sum_{i=1}^h j(x_{n-j+1} - x_{n-j}) = x_n + \dots + x_{n-h+1} - hx_{n-h}$$

and

$$(7.7) \quad \sum_{i=1}^k \sum_{j=1}^i j(x_{n-i+1} - x_{n-i})(x_{n-j+1} - x_{n-j}) = \frac{1}{2} \left\{ \left( \sum_{j=1}^k x_j \right)^2 - \sum_{j=1}^k x_j^2 \right\} \\ = (S_1^2 - S_2)/2.$$

The first of the formulas is obtained by induction and proved for  $h=2, 3$ , etc.. and then, the deduction is easy to get. The second is simply deduced for the developpement of the square of the sum of the  $n-k+1$  numbers. Now, the second term of (7.5) is

$$\sum_{i=1}^k \sum_{j=1}^{i-1} j(x_{n-i+1} - x_{n-i})(x_{n-j+1} - x_{n-j}) + \frac{1}{2} \sum_{i=1}^k i(x_{n-i+1} - x_{n-i})^2 \equiv A + B.$$

Next, using (7.6), on has

$$A = \sum_{i=1}^k (x_{n-i+1} - x_{n-i}) \sum_{j=1}^{i-1} j(x_{n-j+1} - x_{n-j})$$

$$\begin{aligned}
&= \sum_{i=1}^k (x_{n-i+1} - x_{n-i})(x_n + \dots + x_{n-i+2} - (i-1)x_{n-i+1}) \\
&= \sum_{i=1}^k (x_{n-i+1} - x_{n-i})(x_n + \dots + x_{n-i+1} - ix_{n-i+1}) \\
&= \sum_{i=1}^k \sum_{j=1}^i x_{n-i+1}x_{n-j+1} - \sum_{i=1}^k ix_{n-i+1}^2 - \sum_{i=1}^k \sum_{j=1}^i x_{n-i}x_{n-j+1} + \sum_{i=1}^k ix_{n-i}x_{n-i+1} \\
&\equiv A_{11} + A_{12} + A_{21} + A_{22}.
\end{aligned}$$

Now by the change of variables  $i = h - 1$

$$\begin{aligned}
-A_{21} &= \sum_{h=2}^{k+1} \sum_{j=1}^{h-1} x_{n-h+1}x_{n-j+1} = \sum_{h=1}^k \sum_{j=1}^{h-1} x_{n-h+1}x_{n-j+1} + \sum_{j=1}^k x_{n-j+1}x_{n-k} \\
&= \sum_{h=1}^k \sum_{j=1}^h x_{n-h+1}x_{n-j+1} - \sum_{h=1}^k x_{n-j+1}^2 + \sum_{j=1}^k x_{n-j+1}x_{n-k} \\
&= A_{11} - S_2 + x_{n-k}S_1.
\end{aligned}$$

Further

$$2B = \sum_{i=1}^k ix_{n-i+1}^2 + \sum_{i=1}^k ix_{n-i}^2 - 2 \sum_{i=1}^k ix_{n-i}x_{n-i+1} = B_1 + B_2 + B_3$$

with, by change of variable  $i = h - 1$ ,

$$\begin{aligned}
B_2 &= \sum_{h=2}^{k+1} (h-1)x_{n-h+1}^2 = \sum_{h=1}^{k+1} (h-1)x_{n-h+1}^2 = \sum_{h=1}^k (h-1)x_{n-h+1}^2 + kx_{n-k}^2 \\
&= \sum_{h=1}^k hx_{n-h+1}^2 - \sum_{h=1}^k x_{n-h+1}^2 + kx_{n-k}^2 = \sum_{h=1}^k hx_{n-h+1}^2 - S_2 + kx_{n-k}^2.
\end{aligned}$$

Finally

$$B = \frac{1}{2}(-A_{12} - A_{21}) - \frac{1}{2}S_2 + \frac{1}{2}kx_{n-k}^2 - A_{22}$$

and the second term of 7.5 is

$$\begin{aligned}
&A_{11} + A_{12} - A_{11} + S_2 - x_{n-k}S_1 + A_{22} - A_{12} - \frac{1}{2}S_2 + \frac{1}{2}kx_{n-k}^2 - A_{22} \\
&= \frac{S_2 - 2x_{n-k}S_1 + kx_{n-k}^2}{2}.
\end{aligned}$$

This is nothing than the half of

$$\sum_{i=1}^k (x_{n-i+1} - x_{n-k})^2 = \sum_{i=1}^k x_{n-i+1}^2 - 2x_{n-k} \sum_{i=1}^k x_{n-i+1} + kx_{n-k}^2 = S_2 - 2x_{n-k}S_1 + kx_{n-k}^2.$$

This achieves the proof.  $\square$



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LSTA, UPMC, FRANCE AND LERSTAD, UNIVERSITÉ GASTON BERGER DE SAINT-LOUIS, SENE-  
GAL

*E-mail address:* `gane-samb.lo@ugb.edu.sn`, `ganesamblo@ufrsat.org`

*URL:* `www.lsta.upmc.fr/gslo`